LAPLACE TRANSFORM & INVERSE LAPLACE TRANSFORM
LECTURE 8#

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INTRODUCTION

Laplace transforms and Fourier transforms are probably the main two kinds of transforms that are used in engineering problems.

As we will see in later sections we can use Laplace transforms to solve a differential equation and transform it into an algebraic problem.

Laplace transforms can also be used to solve initial and boundary value problem that we can't use any previous method on.

INTRODUCTION

Laplace transforms solve differential equation and corresponding initial and boundary value problems by the followings steps:

1. The given ‘hard’ problem is transformed (Laplace transform) into a ‘simple’ equation.
2. This equation is then solved by purely algebraic manipulation.
3. The obtained equation is transformed back by inverse Laplace transform to obtain the solution of the given problem.

The process is made easier by tables of functions and their Laplace transforms whose role is similar to that of integral tables in calculus.

DEFINITION

Let *f*(*t*) be a given function that is defined for all t≥0. The Laplace transform of *f*(*t*) is defined as:

$$\ell\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Function *F*(*s*) is of the variable *s* which is the Laplace Transform of *f*(*t*).

Furthermore, the origianal function *f*(*t*) is called as inverse Laplace Transform of *F*(*s*) written as:

$$f(t) = \ell^{-1}\{F(s)\}$$

LAPLACE TRANSFORM

Example 1:

Laplace transform of a constant

Let $f(t)=1$ for $t\geq 0$, determine $F(s)$.

Example 2:

Laplace transform of exponential

Let $f(t)=e^{at}$ for $t\geq 0$. Find $\ell\left\{f(t)\right\}=F(s)$

LAPLACE TRANSFORM

THEOREM 1: Linearity of Laplace Transform

For any functions *f(t)* and *g(t)* whose Laplace Transform exist and any constant *a* and *b:*

$$\ell\{a.f(t)+b.g(t)\}=a.\ell\{f(t)\}+b.\ell\{g(t)\}.$$

PROOF

$$\ell\{a.f(t)+b.g(t)\}=\int_0^\infty e^{-st}[af(t)+bg(t)]dt$$

$$=a\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt$$

$$=a.\ell\{f(t)\}+b.\ell\{g(t)\}$$

LAPLACE TRANSFORM

Example 3:

Laplace transform of hyperbolic function

Let $f(t)=\cosh at=(e^{at}+e^{-at})/2$. Find $\ell\{f(t)\}=F(s)$

Solution:

$$F(s)=\frac{1}{2}\ell\{e^{at}\}+\frac{1}{2}\ell\{e^{-at}\}$$

$$\vdots$$

$$=\frac{s}{s^2-a^2}$$

LAPLACE TRANSFORM

Example 4: cosine and sine functions

$\ell\{\cos \omega t\}$ and $\ell\{\sin \omega t\}$

can be found from Euler equation:

$$\Rightarrow e^{j\omega t} = \cos \omega t + j \sin \omega t \qquad \text{where}: j = \sqrt{-1}$$

$$\ell\{e^{j\omega t}\} = \int_0^\infty e^{j\omega t} e^{-st} dt = \int_0^\infty e^{(j\omega - s)t} dt = \frac{1}{s - j\omega}$$

$$= \frac{1}{s - j\omega} \cdot \frac{s + j\omega}{s + j\omega} = \frac{s}{s^2 + \omega^2} + j \frac{\omega}{s^2 + \omega^2}$$

$$= \ell\{\cos(\omega t)\} + j.\ell\{\sin(\omega t)\}$$

LAPLACE TRANSFORM

THEOREM 2: First Shifting Theorem (s-shifting)

If *f(t)* has transform *F*(*s*), then $e^{at}f(t)$ has the transform *F*(*s-a*):

$$\ell\left\{e^{at}f(t)\right\} = F(s-a)$$

Or if we take inverse:

$$e^{at}f(t) = \ell^{-1}\left\{F(s-a)\right\}$$

PROOF

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$\Rightarrow F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\,dt = \int_0^\infty e^{at}f(t)e^{-st}\,dt$$

$$= \ell\left\{e^{at}f(t)\right\}$$

LAPLACE TRANSFORM

Example 5:

Evaluate the Laplace transform of the followings:

$$f(t) = e^{at}\cos\omega t$$

$$f(t) = e^{at}\sin\omega t$$

Solution

$$\ell\left\{e^{at}\cos\omega t\right\} = \frac{s-a}{(s-a)^2+\omega^2} \quad \text{since} \quad \ell\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$\ell\left\{e^{at}\sin\omega t\right\} = \frac{\omega}{(s-a)^2+\omega^2} \quad \text{since} \quad \ell\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

LAPLACE TRANSFORM

Example 6:

Laplace transform of $f(t)=t$.

$$\ell\{t\} = \int_0^\infty te^{-st}\,dt$$

(integral by parts)

$$\ell\{t\} = \frac{1}{s^2}$$

LAPLACE TRANSFORM

Example 7:

Laplace transform of $f(t)=t^2$.

$$\ell\left\{t^2\right\}=\int_0^\infty t^2 e^{-st}\,dt$$

$$\text{(integral by parts)}$$

$$\ell\left\{t^2\right\}=\frac{2}{s^3}=\frac{2!}{s^3}$$

In general:

$$\ell\left\{t^n\right\}=\frac{n!}{s^{n+1}}$$

LAPLACE TRANSFORM

Example 8:

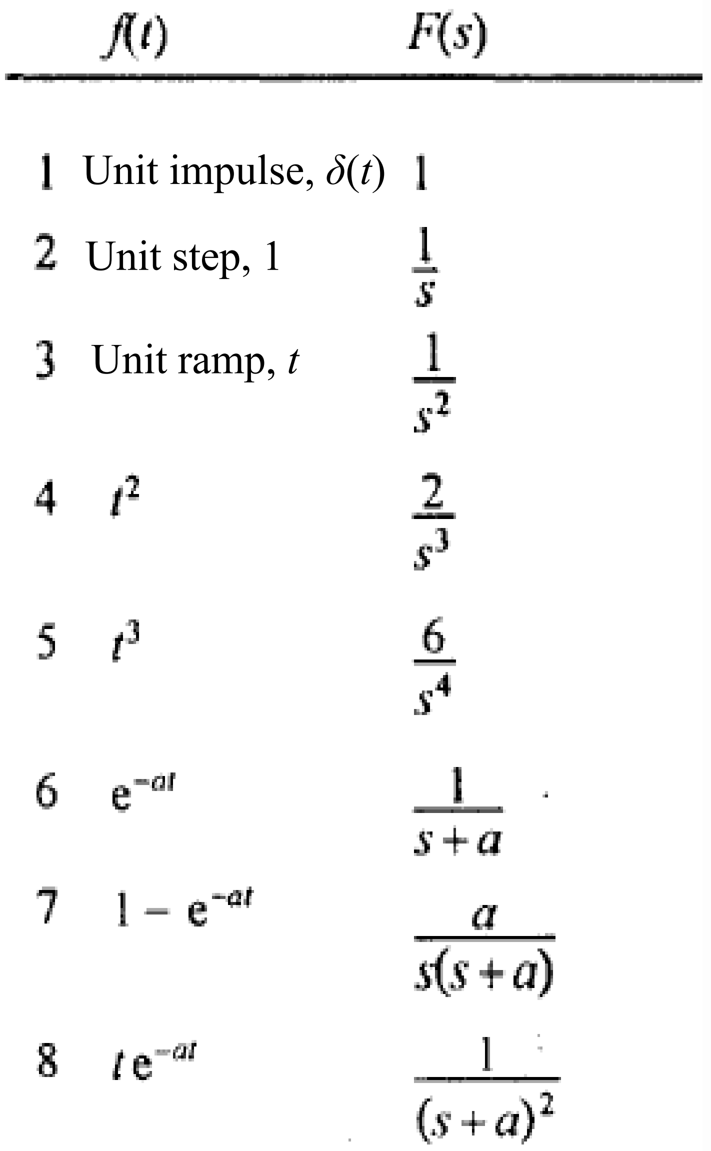
Laplace transform of $f(t)=e^{at}t^{n}$.

Using the first shifting theorem:

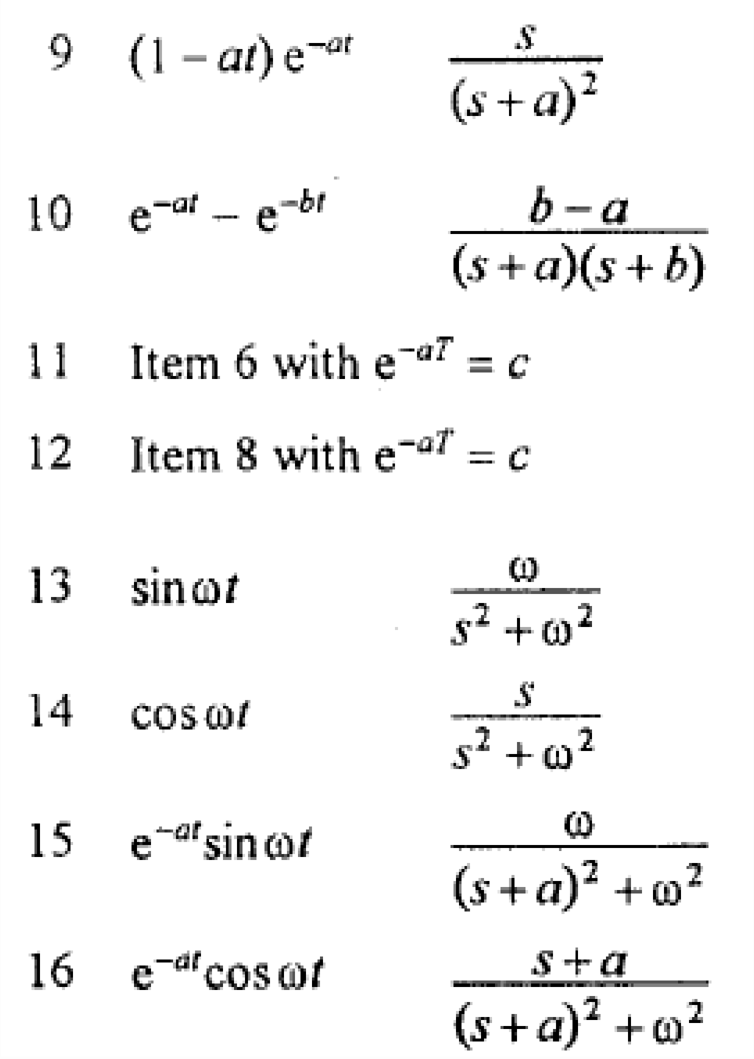
$$\ell\left\{e^{at}t^{n}\right\}=\frac{n!}{(s-a)^{n+1}}$$

LAPLACE TRANSFORM

Laplace Transform of Some Functions



	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step, 1	$\frac{1}{s}$
3	Unit ramp, t	$\frac{1}{s^2}$
4	t^2	$\frac{2}{s^3}$
5	t^3	$\frac{6}{s^4}$
6	e^{-at}	$\frac{1}{s+a}$
7	$1-e^{-at}$	$\frac{a}{s(s+a)}$
8	te^{-at}	$\frac{1}{(s+a)^2}$



9	$(1-at)\,e^{-at}$	$\frac{s}{(s+a)^2}$
10	$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
11	Item 6 with $e^{-aT}=c$	
12	Item 8 with $e^{-aT}=c$	
13	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
14	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
15	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
16	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

INVERSE LAPLACE TRANSFORM

Example 9-11: Find inverse Laplace transform of the followings…

9. $$\frac{7}{(s-1)^3}$$

10. $$\frac{s-6}{(s-1)^2+4}$$

Solution:

9. $$\frac{7}{2}t^2e^t$$

10. $$e^t\cos 2t-\frac{5}{2}e^t\sin 2t$$

DERIVATIVE OF TRANSFORMS

THEOREM 3: Derivatives of Transforms

If *f(t)* has transform *F*(s), and n=1,2,3,… then:

$$\ell\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

PROOF (for n=1)

$$\ell\left\{t.f(t)\right\} = -\frac{d}{ds} F(s)$$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{d}{ds}\left[e^{-st} f(t)\right]dt = -\int_0^\infty e^{-st} t.f(t)dt = -\ell\left\{t.f(t)\right\}$$

(assuming that interchanging of integration w.r.t to *t* and differentiation w.r.t. to *s* is possible)

Example 11:

Evaluate the Laplace transform of the followings:

$$f(t) = e^{at} \cos \omega t$$

$$f(t) = e^{at} \sin \omega t$$

Example 12-14:

Evaluate the Laplace Transform of the followings by applying the derivatives of transforms theorem above:

12. $\ell\left\{te^{3t}\right\}$

13. $\ell\left\{t\sin\omega t\right\}$

14. $\ell\left\{te^{-t}\cos\omega t\right\}$

Solution:

12. $\dfrac{1}{(s-3)^2}$

13. $\dfrac{2\omega s}{(s^2+\omega^2)^2}$

14. $\dfrac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2}$